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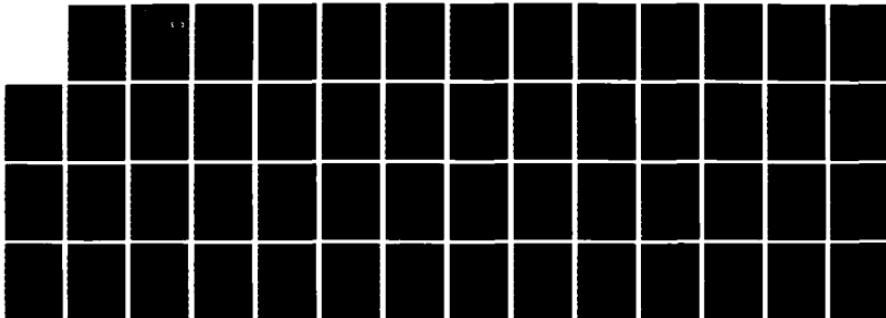
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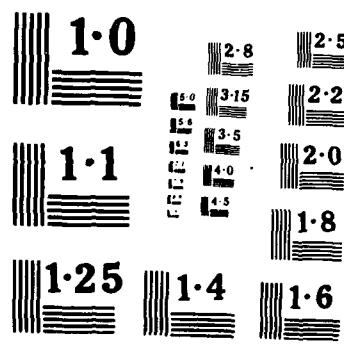
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An Expert-System-Like Feedback Approach  
in the hp-Version of the Finite Element Method

I. Babuska  
Institute for Physical Science and Technology  
University of Maryland

Ernest Rank  
Institute for Physical Science and Technology  
University of Maryland

and

Fachgebiet Elektronisches Rechnen im Konstruktiven Ingenieurwesen  
Technische Universität München  
Federal Republic of Germany

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An Expert-System-Like Feedback Approach  
in the hp-Version of the Finite Element Method

I. Babuška<sup>1</sup>

Institute for Physical Science and Technology  
University of Maryland

Ernst Rank<sup>2</sup>

Institute for Physical Science and Technology  
University of Maryland

and

Fachgebiet Elektronisches Rechnen im Konstruktiven Ingenieurbau  
Technische Universität München  
Federal Republic of Germany

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Abstract

It has been theoretically shown that exponential rate of convergence in the energy norm can be obtained by a proper combination of the h- and p-extension of the finite element method for essentially all engineering problems in plane elasticity, including those with singularities in the exact solution. In this paper we present an expert system frame, which guides the user to an optimal selection of the mesh and degree distribution. After a crude preliminary computation the expert system predicts the performance of various mesh and degree combinations and provides the analyst with rational support for his decisions about the mesh design. Numerical examples show that the suggested approach gives superior results with the minimum effort in human data preparation and computer time.

Keywords: Finite element methods, hp-version of the finite element method; hp-extension, expert systems, elasticity, adaptive methods; error estimations. ↙



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## 1. INTRODUCTION

Today, powerful and sophisticated methods in computational mechanics are widely available. Yet, these tools are very often not used as efficiently and intelligently as they could be. The reason is that the more complex and general this software is, the more expertise, i.e., experience and knowledge of the user, is necessary to use all its possibilities efficiently. In general, it is impossible for a single user to have all this expertise.

Traditionally an expert's knowledge has been made available through direct consultation or by his training of a group of apprentices. Both approaches are slow and expensive and have obvious limitations.

Today's technology of building knowledge-based systems offers an opportunity to change these approaches by capturing the desired expertise and storing it for use in a computer system so that newly acquired knowledge can easily be incorporated and efficiently disseminated.

The frame of an expert system belongs to the field of artificial intelligence where vast literature is available. (see, e.g., [1], [2].) This frame has been applied more or less successfully in various fields. For example, we refer to [3] for a survey of literature related to engineering problems, and to [4] for the use of an expert system helping to select particular numerical methods. Various conferences and symposia have recently addressed this type of question in different contexts.

One of the main parts of the successful design of an expert system is to formulate (and justify) the rules, heuristics, experience (in general expertise) which can be incorporated into the expert system frame. The formulation of these rules in the context of an optimal mesh design for finite element analysis will be the topic of this paper. We will also utilize the ideas of

adaptive approaches, and we will consider especially the problem of the optimal use of the possibilities of  $h$ -,  $p$ - and  $hp$ -extension when the goal is to get the solution in the range of a prescribed tolerance of the error in the energy norm. In a very basic sense, adaptive procedures and reliability assessments (e.g., a-posteriori estimates) have clear connection to the expert system area although they have only algorithmic character. (For a survey of basic achievements in this area and literature we refer to [5]).

Any finite element analysis can be essentially separated into two parts, the decision phase and the execution phase which carries out the decisions, although these phases are often interwoven.

The decision phase includes the specification of the goals of the analysis, (e.g., computation of stresses in some regions, computation of the stress intensity factors, frequencies of vibration, etc.), specification of the available information and its reliability (as the domain loads, the material properties, etc.), the desired accuracy (as used norms of the error, acceptable tolerances) and the available resources of computer time and storage. Based on these specifications, the user selects the mathematical model, constructs the mesh using available mesh generators, selects the methodology of computation of the desired data (e.g. elements and degrees). He selects the method of interpreting the results (for example graphical postprocessing) so that human and computer cost are expected to be as small as possible. However, most finite element programs and mesh generators provide little rational support for proper decisions and obviously expertise is needed for an optimal use of these programs.

There are three basic versions of the finite element method, the  $h$ -,  $p$ - and  $hp$ -version. In the classical  $h$ -version the polynomial degree  $p$  is fixed

at low levels, typically  $p=1,2$ , and the accuracy is achieved by refining the mesh. The  $p$ -version fixes the mesh while achieving accuracy by increasing the degrees of the elements. The  $hp$ -version is a combination of both. The theory of the  $p$ -, and  $hp$ -version is given, for example, in [6], [7] [8] and [9], and for practical applications we refer to [10] and [11]. The  $p$ - and  $hp$ -versions are recent developments and, besides some research codes, the authors know of only two commercial programs based on the  $p$ -version. These are the computer program PROBE (Noetic Technology, St. Louis, USA) [12] for solving two-dimensional problems (with  $p=1,\dots,8$ ) and FIESTA (ISMES, Bergamo, Italy) [13] for three-dimensional problems ( $p=1,\dots,4$ ).

One of the main features of the  $hp$ -version is that in practically all engineering cases (including domains with corners, cracks, etc.) exponential rate of convergence can be achieved (see [8]).

Although theoretically the method is reasonably understood today, there is still the practical problem of how to design the mesh and the degree of the elements so that the given accuracy is obtained in the optimal way. The mesh design and the assignment of the degree of the elements can be based on an adaptive approach. We refer to [7] for the analysis in a one-dimensional setting.

The approach discussed in this paper is essentially different from the adaptive methods, although there are, necessarily, similarities to the aims of the adaptive codes. Our approach clearly separates the phases of decision and execution and adds the part of the advise and communication with the user (utilizing ideas of knowledge-based systems). It uses well-defined mathematical rules, heuristics and experience to guide the analyst to the design of nearly optimal meshes and degree of the elements leading to the desired accuracy. The main idea of the approach presented here is that after the

design of the most crude mesh modelling the geometry (which always has to be made by user's interference) one crude (cheap) computation is made to obtain the main characteristics of the solution. Using this information, the user obtains nearly optimal meshes and degree of elements (for the desired accuracy) together with information about the relationship of computer cost and predicted accuracy.

Based on the user's response, the nearly optimal mesh and degree of elements is designed in one step (and not, as in the adaptive codes mentioned above, in a subsequent mesh refinement). The solution is then obtained in the execution phase, which is typically about 80 to 90% of the total cost.

The knowledge-based system ideas we are using in our approach

- a) help to identify the class to which the concrete problem belongs,
- b) characterize the critical areas of the solution (heuristics),
- c) give the user concrete ways for extracting essential information about the problem (algorithms),
- d) give the user qualitative information about predicted features of various optimal combinations of meshes and p-distribution and the relation between the cost and accuracy (heuristics and algorithm),
- e) use the analyst's decision to construct (automatically) the mesh, p-distribution, etc., and present the solution to the user (algorithm) with a-posteriori assessment of the reliability (algorithm, heuristics) of the solution which gives him the choice of accepting or rejecting the result.

In this paper we will restrict ourselves to the class of elasticity problems on polygonal domains with homogeneous and isotropic material and smooth boundary conditions, although the basic concepts above can be applied to more general situations.

## 2. THE ELASTICITY PROBLEM ON A POLYGONAL DOMAIN AND ITS BASIC PROPERTIES

Let us consider the plane (strain) elasticity problem on a polygonal domain  $\Omega$  shown in Fig. 2.1.

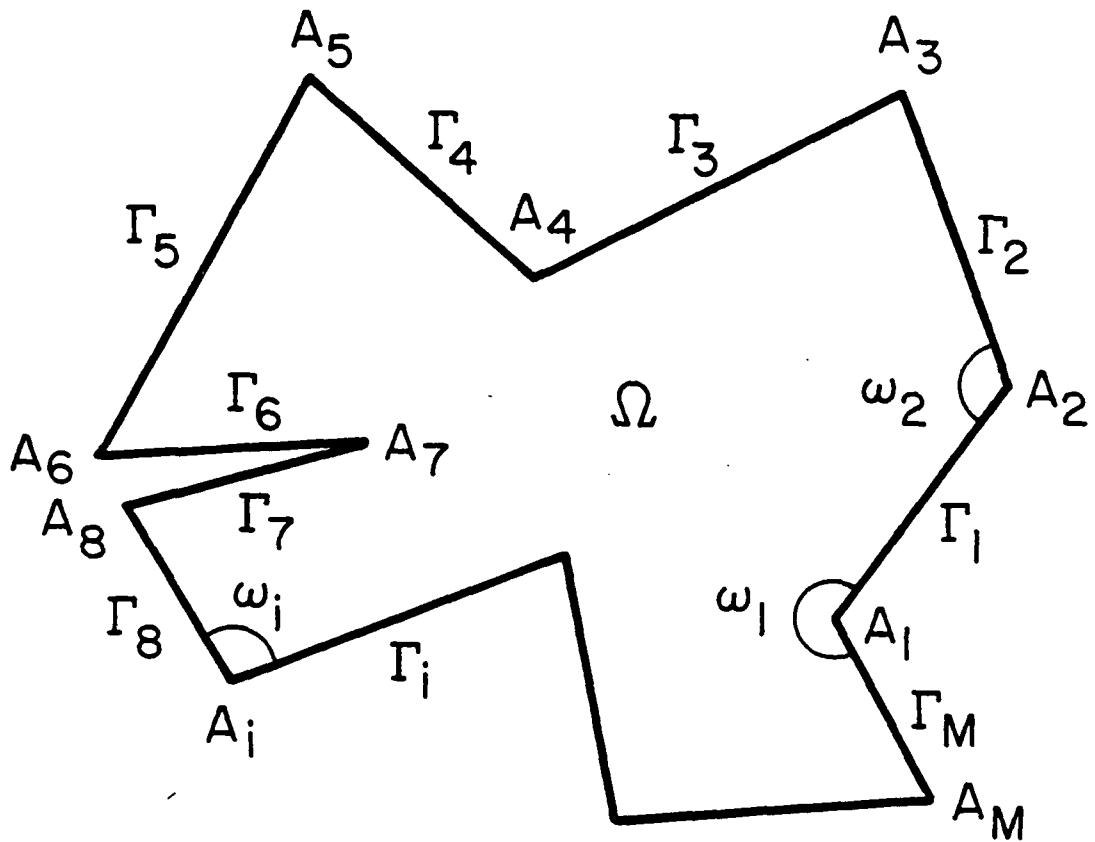


Figure 2.1. Polygonal domain.

By  $\partial\Omega$  we denote the boundary of  $\Omega$  and have

$\partial\Omega = \bigcup_{i \in \gamma} \bar{\Gamma}_i$ ;  $\gamma = \{1, \dots, M\}$  where  $\Gamma_i$  are the open edges connecting the vertices  $A_i, A_{i+1}$  ( $A_{M+1} = A_1$ ). By  $\omega_i$  we denote the internal angles of  $\Omega$  in  $A_i$ . We will not exclude the case  $\omega_i = 2\pi$  and  $\omega_i = \pi$ .

Let  $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ ;

$$\Gamma_D = \bigcup_{i \in D \subset \gamma} \bar{\Gamma}_i ; \quad \Gamma_N = \partial\Omega - \bar{\Gamma}_D .$$

$\Gamma_D$  will be called displacement and  $\Gamma_N$  traction boundary. We allow

$$\Gamma_D = \partial\Omega \quad \text{or} \quad \Gamma_D = \emptyset.$$

Considering the plane (strain) elasticity problem on  $\Omega$ , we denote by  $u = (u_1, u_2)^T$  the displacement vector,  $E$ ,  $0 \leq v < 1/2$  the Young's elasticity module and Poisson's ratio, respectively. The strain energy is given by

$$(2.1) \quad W(u) = \frac{E}{2(1-v)(1+v)} \int_{\Omega} \left[ (1-v) \left( \frac{\partial u_1}{\partial x_1} \right)^2 + 2v \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} \right. \\ \left. + (1-v) \left( \frac{\partial u_2}{\partial x_2} \right)^2 + \left( \frac{1-2v}{2} \right) \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)^2 \right] dx$$

Let  $E = \{u \mid W(u) < \infty, u = 0 \text{ on } \Gamma_D\}$  be the energy space and let  $\|u\|_E^2 = W(u)$ . It is well known that the energy space  $E$  is equivalent to the Sobolev space  $H^1(\Omega)$  (modulo rigid body motion displacement functions when  $\Gamma_D = \emptyset$ ).

The elasticity problem (without volume forces) for prescribed tractions  $T = (t_1, t_2)^T$  on  $\Gamma_N$  now has unique (weak) solution  $u_0$  which minimizes over  $E$  the quadratic functional

$$(2.2) \quad \Pi(u) = W(u) - \sum_{i=1}^2 \int_{\Gamma_N} t_i u_i ds$$

If  $\Gamma_D = \emptyset$  then we assume that the tractions satisfy the usual equilibrium condition. Furthermore, we assume that  $t_i$ ,  $i=1,2$  are analytic on every (closed) edge  $\Gamma_j \subset \Gamma_N$ . The solution  $u_0$  is then analytic on  $\overline{\Omega} - \bigcup_{i=1}^M A_i$  and for some  $0 < \beta < 1$ ,  $d > 1$ ,  $c > 0$ ,  $|\alpha| \geq 2$ .

$$(2.3) \quad \left[ \int_{\Omega} (D^\alpha u_{0,i})^2 [\phi]^{2(|\alpha|-2+\beta)} d\Omega \right]^{1/2} \leq C_d |\alpha| |\alpha|!$$

holds for all  $\alpha = (\alpha_1, \alpha_2)$ , where  $\alpha_i \geq 0$  integer,  $\alpha_1 + \alpha_2 = |\alpha|$  with

$\Phi = \prod_{i=1}^M \|x - A_i\|$ .  $\|x - A_i\|$  is the Euclidean distance of  $x$  to  $A_i$ , and

$D^{\alpha} u = \frac{\partial^{\alpha_1 + \alpha_2} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$ . Further, in the neighborhood

$S_i = \Omega \cap \{x \mid \|x - A_i\| < r_0\}$  of the vertices  $A_i$  the exact solution  $u_0$  can be written in the form

$$(2.4) \quad u_0 = \sum_{j=1}^{Q_i} K_{ij} \phi_{ij}(r_i) g_{ij}(\theta_i) + w_{Q_i, i}$$

where  $(r_i, \theta_i)$  are polar coordinates with respect to  $A_i$ ,  $\phi_{ij}(r_i)$ ,  $g_{ij}(\theta_i)$  are a-priori known functions independent of  $u_0$  (depending on  $w_i$ ) and  $K_{ij}$  (scalars) are stress intensity factors (dependent on  $u_0$ ) and  $w_{Q_i, i}$  is an analytic function on  $\bar{S}_i - A_i$  which is smoother than the first term on the right-hand side of (2.4).  $Q_i$  is a positive integer which will be chosen for our practical purposes as 1 or 2. For more details, see [14], [15] and [16].

The function  $g_{ij}(\theta_i)$  is analytic up to the boundary of  $S_i$  and  $\phi_{ij}(r_i) = \operatorname{Re} (r_i^{\alpha_{ij}} \lg^{\rho} r_i)$ ;  $\rho \geq 0$  integer.  $\alpha_{ij}$  are in general complex numbers with positive real parts and  $\operatorname{Re} \alpha_{i,j+1} \geq \operatorname{Re} \alpha_{ij}$ . Hence the first terms in (2.4) are the most singular ones. For the way to compute stress intensity factors we refer to [17], and [18].

Table 2.1 shows the values of  $\alpha_{i1}$  and  $\alpha_{i2}$  for some internal angles  $\omega_i$  when the tractions are prescribed on both sides of  $A_i$ . In this case  $\rho = 0$ .

$\omega_i$	$\alpha_{i1}$	$\alpha_{i2}$
360°	0.500000	0.500000
270°	0.544484	0.908529
240°	0.615731	1.148913
225°	0.673583	1.302086
210°	0.751975	1.485812

Table 2.1. Exponents  $\alpha_{i1}$  and  $\alpha_{i2}$  for selected internal angles  $\omega_i$  at a node  $A_i$ .

### 3. THE HP-VERSION AND ITS BASIC PROPERTIES

As was stated in the introduction, the hp-version is a proper combination of the h- and p-version of the finite element method. For simplicity we will assume that the polynomial degree of all the elements is the same. It has been shown [8] that if the exact solution satisfies (2.3) then for a proper mesh  $\Delta(p)$  depending on  $p$  the error of the finite element solution  $u_{FE}$  measured in the energy norm decays exponentially. More precisely

$$(3.1) \quad \|e\|_E = \|u_{FE} - u_0\|_E \leq C e^{-a N^{1/3}(p)} \quad p \rightarrow \infty$$

where  $a > 0$  and  $N$  is the number of degrees of freedom.

The error in particular elements has two essential parts. The local one depends only on the solution in the given element and the global one reflects the influence of the error in the entire domain (so-called pollution error). The global error is (for properly designed meshes) smaller than the local one. If an element is not adjacent to any vertex, then the local error of the (p-version) finite element method is of the order  $e^{-a N_1^{1/2}}$  where  $N_1$  is the number of degrees of freedom associated with the element whereas the error is of order  $N_1^{-\beta}$  when the particular element has a vertex in a corner of the domain. Hence the decay of the error is more rapid in areas far from corners of the domain than in the neighborhood of the corners. Therefore only proper refinement in the neighborhood of corners is needed, while in the elements not adjacent to corners high accuracy can be achieved only by increasing  $p$ . For practical engineering accuracies we can treat convex corners as no corners (provided that the boundary condition is the same on both sides of the vertex).

So the proper mesh  $\Delta(p)$  is a geometric one with geometric refinement in the neighborhood of every reentrant corner. An example of such a mesh with 2 layers at  $A_0$  is shown in Fig. 3.1.

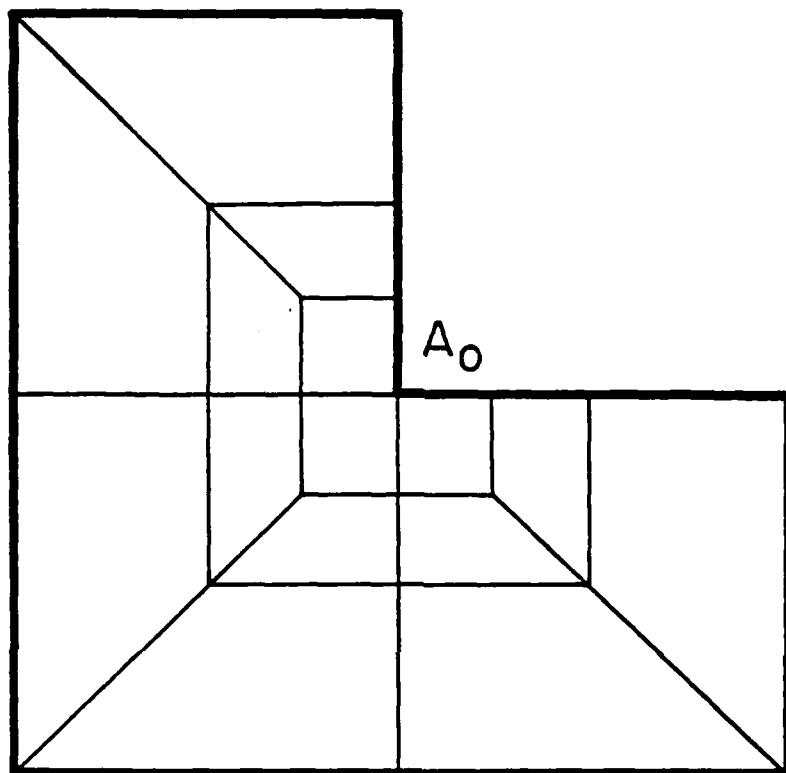


Figure 3.1. Example of a geometric mesh with two layers.

It has been shown that the optimal number  $n$  of layers of elements increases with the polynomial degree  $p$ , that the optimal ratio of the geometric mesh is independent of the strength of the singularity,  $p$ , and the number  $n$  of layers and has a magnitude of .15. (see [7] and [8].) Thus a very strong grading towards the singularity is obtained.

If the mesh is fixed (with different numbers of layers) and the polynomial degree  $p$  increases, then the error behaves as schematically shown in Fig. 3.2.

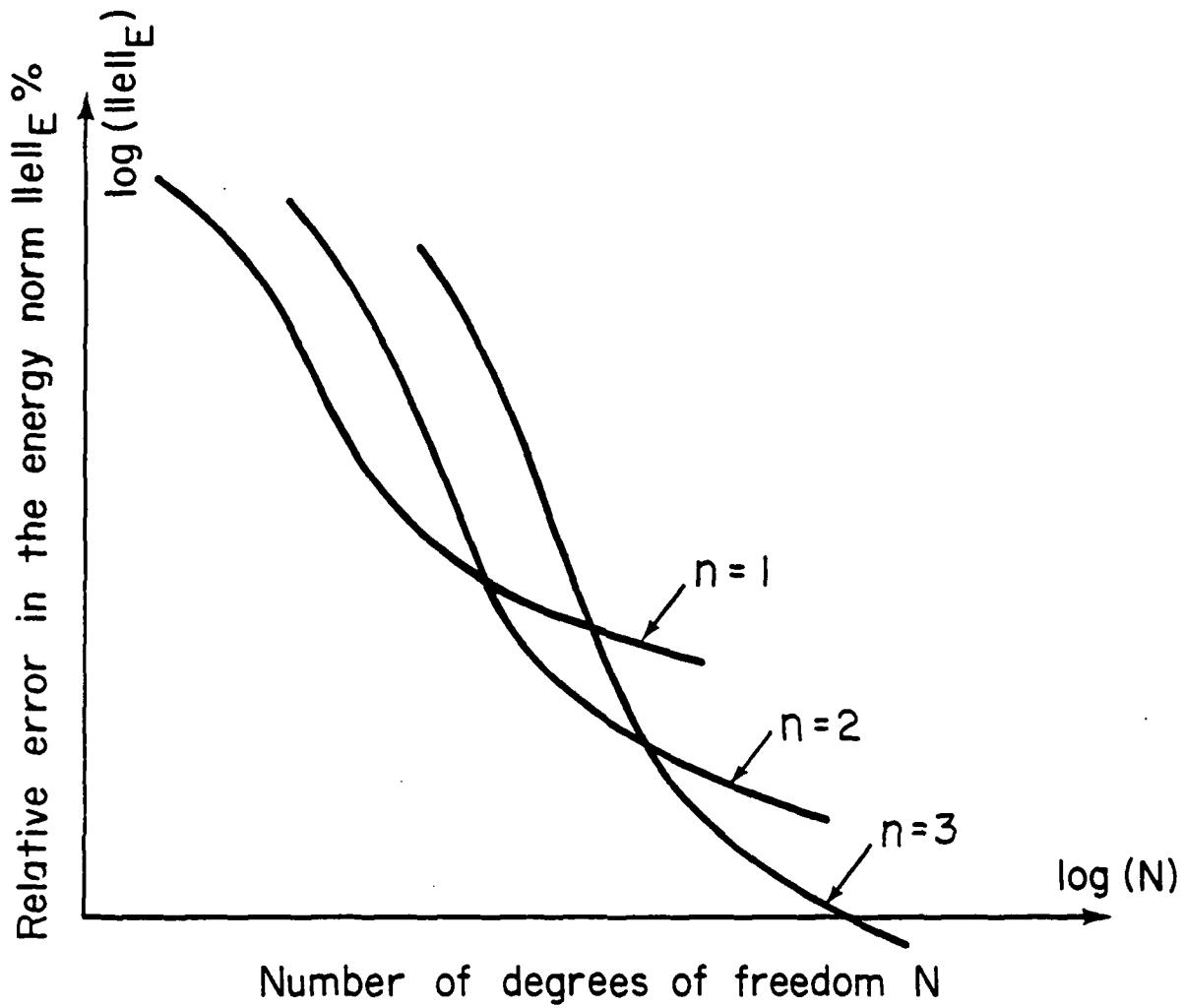


Figure 3.2.  $\|e\|_E$  in dependence of the number of degrees of freedom for different number of layers  $n$ .

For each fixed number of layers we see a typical reverted S-curved behaviour of the error. We see the preasymptotic phase, (curved down), when the error decays exponentially and the asymptotic phase, (straight line), when the error decays algebraically. For theoretical results we refer to [7] and [8].

Figure 3.2 shows that at reentrant corners for high number of layers and low polynomial degree  $p$  the mesh can be overrefined (with respect to  $N$ ) while an insufficient number of layers leads to an underrefined mesh. The

detailed behaviour is complicated but it strongly depends on the values of  $K_{ij}$  and  $\alpha_{ij}$  in (2.4).

For practical reasons we wish to design a mesh (i.e. the number of layers in every corner) depending on  $p$  so that the desired accuracy can be achieved in a most economic way.

The definition of the cost of a computation can vary. It can, for example, be a function of the degree of freedom  $N$  or the number of operations required in the computation and elimination of the stiffness matrix, depending on the sparsity of the matrix.

Our experience shows that (because of overhead, 10) it is enough to consider the cost as a function of  $N$  only.

#### 4. THE MAIN PRINCIPLES OF THE MESH DESIGN

The main goals for the optimal mesh design are the following:

- 1) The user supplies only the basic data which have to be kept to a minimum.
- 2) The user should get rational support for his finite element analysis, mesh design, relationship between accuracy and cost of the computation, etc., and he should be given the opportunity to make the crucial decisions in the entire process. The expert system provides, then, the support for the user to execute his decisions automatically.

The finite element analysis based on the expert system's support has the following components:

- a) Interactive graphical mesh generation and data characterization

The finite element process starts with the characterization of the problem by definition of the domain by a basic mesh which is as coarse as possible (or by solid modeller information) of the boundary conditions, the loads, material properties, etc. Graphical input should be used wherever possible and analysis of the data for possible input errors should be provided.

In this paper we will concentrate only on the aspects of the domain definition and we will assume that they belong to the class of polygons. The basic mesh will also serve as the basic set of elements which will be dealt with.

- b) Basic decisions about critical and noncritical elements of the basic mesh

An element is said to be noncritical if the optimal rate of convergence (in the range of engineering accuracy) can be obtained by increasing the

degree of the elements without any refinement. An element is said to be critical if the optimal rate of convergence cannot be achieved only by increasing the degree, but the element has to be refined. The decision about these two classes has to be based on the expertise which uses theoretical results, heuristics and experience.

For example, all elements adjacent to reentrant corners or vertices with changing boundary conditions as well as elements with excessive aspect ratio or elements where the load is unsmooth are critical. We will assume in this paper that the critical elements of the basic mesh are only those adjacent to vertices with concave internal angles.

c) Extraction of critical data about the solution

Crude preliminary computation and various extraction procedures give the basic qualitative and quantitative information about the solution in critical areas. Especially in this paper, the approximate determination of the stress intensity factors in (2.4) will give essential information about the solution. We will extract the stress intensity factors by the procedures described in [17] and [18].

d) Prediction of the performance of various optimal mesh and degree combinations

This prediction is one of the critical parts of the system and will be elaborated on in detail in section 5. The prediction characterizes the performance of the meshes with different numbers of refinement layers  $n$  in the critical elements in dependence of the polynomial degree  $p$ . This prediction utilizes the information obtained by the extraction technique under c) above.

e) Presentation of the predicted performance for the user's decision

Based on the analysis in d), the basic data about the predicted relation between cost and accuracy are graphically presented to the user. Other characterizations may also be provided. For example, in the case of multiple loads the user should get advice on whether it is advantageous to solve the problem with one mesh or to use different meshes for various classes of loads. In this paper we will restrict ourselves to the case of one load only.

f) Mesh generation for the (final) computation based on the user's decision

Based on the user's decision, the mesh which is expected to give acceptably accurate results in an optimal way (for given optimal degree of the elements) is constructed automatically.

g) Finite element computation and assessment of the reliability of the computed data

The finite element code uses the optimal mesh and determined degree of the elements. It provides the solution and a-posteriori assessment of the reliability of the basic computed data (e.g., error in energy norm). In this paper we will use the code PROBE for the computation.

h) Presentation of the basic results of the computation for the user's acceptance

The basic results obtained under g) are presented to the user and with the possible assistance of the expert system the user decides whether the results are acceptable.

i) Postprocessing and visualization of the results

If the user accepts the finite element results, various postprocessing procedures are appended to give him the results in which he is interested, to provide graphical display, etc. The expert system will probably help to interpret the results.

j) Processing of rejected results

If the user rejects the results because the prediction was not accurate enough, then the computed data are used analogously to the crude mesh data - with additional information and the process is repeated from c). Rejection should occur only under very exceptional circumstances.

The underlying data structure of the whole process is a project data base where the complete history of the analysis is stored. This makes it easily possible to add new load cases or to modify the needs of the accuracy of the results.

5. PREDICTION OF THE PERFORMANCE OF VARIOUS OPTIMAL MESHES AND DEGREES FOR CRITICAL ELEMENTS

5.1 Error analysis in critical elements

We have seen that in the neighborhood  $S_i$  of the vertex  $A_i$  the solution has the form (2.4). Because the function  $g_{i,j}(\theta)$  is smooth, the behaviour of the error is qualitatively the same as if  $g_{i,j}(\theta) = 1$ . Hence we can expect that the error in the sector  $S_i$  is essentially the same as in the one-dimensional setting in the interval  $(0, r_0)$  with the weight  $x$  expressing polar coordinates. For the two-dimensional analysis we refer to [8] which justifies our assumptions.

Let us first study the problem of the best approximation on the interval  $I = (-1, 1)$ . Let for  $\xi < 1$

$$(x - \xi)_+ = \begin{cases} x - \xi & \text{for } x > \xi \\ 0 & \text{for } x \leq \xi \end{cases}$$

and let  $\psi(\alpha, \xi, x) = (x - \xi)_+^\alpha$ .

Denote

$$(5.1) \quad E_p(\alpha, \xi) = \left[ \inf_{\omega} \int_{-1}^{+1} (\psi - \omega)^2 dx \right]^{1/2}$$

where the infimum is taken over the set of all polynomials of degree  $\leq p$ .

Analogously let

$$(5.2) \quad E_p^*(\alpha, \xi) = \left[ \inf_{\omega} \int_{-1}^{+1} (1 - x^2)(\psi - \omega)^2 dx \right]^{1/2}.$$

The values  $E_p(\alpha, \xi)$  and  $E_p^*(\alpha, \xi)$  express the minimal error achievable by the polynomial approximation. Now we have

Theorem 1 :

a) If  $\xi = -1$  then

$$(5.3) \quad E_p(\alpha, -1) = C_0(\alpha) \frac{2^{\alpha + \frac{1}{2}}}{(p+1)^{2\alpha+1}} \left(1 + O\left(\frac{1}{p}\right)\right)$$

where  $C_0(\alpha) = \frac{(\Gamma(1+\alpha))^2 |\sin(\pi\alpha)|}{\pi \sqrt{2\alpha+1}}$

b) If  $\xi < -1$  then

$$(5.4) \quad E_p(\alpha, \xi) = C_1(\alpha) \left(\frac{1-r^2}{2r}\right)^\alpha \frac{r^{p+1}}{(p+1)^{\alpha+1}} \left(1 + O\left(\frac{1}{p}\right)\right)$$

where  $\zeta > 0, r = \frac{1}{|\xi| + \sqrt{\xi^2 - 1}}$

and  $C_1(\alpha) = \frac{\Gamma(1+\alpha) |\sin(\pi\alpha)|}{\sqrt{\pi}}$

$\Gamma(\alpha)$  is the gamma function and  $O\left(\frac{1}{p}\right)$  is a function which is of the same order as  $\frac{1}{p}$ . For the proof, see [7] theorems 5 and 6.

We see that for  $\xi = -1$ , i.e., when the singularity is in the vertex, the error decays algebraically with the order  $(p+1)^{2\alpha+1}$ , while when  $\xi < -1$ , i.e., the singularity is outside the interval, the decay is exponential.

A similar theorem holds for  $E_p^*$  :

Theorem 2 :

a) If  $\xi = -1$  then

$$(5.5) \quad E_p^*(\alpha, -1) = C_2(\alpha) \frac{2^{\alpha+1}}{(p+1)^{2\alpha+2}} \left(1 + O\left(\frac{1}{p}\right)\right)$$

where

$$C_2(\alpha) = \frac{(\Gamma(\alpha+2))^2 |\sin(\pi\alpha)|}{\pi(\alpha+1)^{3/2}} .$$

b) If  $\xi < -1$  then

$$(5.6) \quad E_p^*(\alpha, \xi) = C_3(\alpha) (1-r^2) \left(\frac{1-r^2}{2r}\right)^\alpha \frac{r^{p+1}}{(p+1)^{\alpha+1}} \left(1+O\left(\frac{1}{p}\right)\right)$$

where  $\xi > 0$ ,  $r$  is defined as in theorem 1 and

$$C_3(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\alpha+2) |\sin(\pi\alpha)|}{\sqrt{\pi}}$$

Theorem 2 can be easily obtained by modifying the proof of theorem 1, using the fact that

$$\int_{-1}^{+1} (1-x^2)^{\alpha} P_j P_k dx = \begin{cases} 0 & \text{for } j \neq k \\ j(j+1) \int_{-1}^{+1} P_j^2 dx & \text{for } j = k \end{cases}$$

where  $P_j$  are the Legendre polynomials.

Comparing (5.3) and (5.5) we see that  $E_p^*$  is smaller than  $E_p$  and that  $E_p^* \rightarrow 0$  for  $\alpha > -1$  while  $E_p \rightarrow 0$  only for  $\alpha > -1/2$ .

On the other hand, comparing (5.4) and (5.6) we see that  $E_p$  and  $E_p^*$  are up to a constant essentially the same, (i.e.,  $E_p = E_p^*$ ).

Let us now define

$$E_p^{**}(\alpha, \xi) = \inf_{\omega} \left[ \int_{-1}^{+1} (1+x)(\psi-\omega)^2 dx \right]^{1/2} .$$

As  $E_p^* = E_p$  for  $\xi < -1$ , it is clear that in this case also  $E_p^* = E_p^{**}$ . Moreover,  $E_p^* = E_p^{**}$  also in the case  $\xi = -1$ , since near the singularity  $-1$  of  $\psi$  the weights  $1+x$  and  $1-x^2$  are essentially the same.

So far we have been interested in the case  $I=(-1,1)$ . Let us now take  $I=(a,b)$  and

$$\psi = (x - d)_+^\alpha \quad d < a$$

Defining

$$E_p(a,b,d,\alpha) = \left[ \inf_{\omega} \int_a^b (\psi - \omega)^2 dx \right]^{1/2}$$

and

$$E_p^*(a,b,d,\alpha) = \left[ \inf_{\omega} \int_a^b (x-a)(b-x)(\psi-\omega)^2 dx \right]^{1/2}$$

$$E_p^{**}(0,b,0,\alpha) = \left[ \inf_{\omega} \int_0^b x(\psi - \omega)^2 dx \right]^{1/2}$$

we easily get the relationship between  $E_p(\alpha, \xi)$  and  $E_p(a, b, d, \alpha)$  by a scaling argument.

Lemma 3:

$$(5.7a) \quad E_p(a,b,d,\alpha) = h^{\alpha+1/2} E_p^*(\xi, \alpha)$$

$$(5.7b) \quad E_p^*(a,b,d,\alpha) = h^{\alpha+3/2} E_p^*(\xi, \alpha)$$

$$(5.7c) \quad E_p^{**}(0,b,0,\alpha) = h^{\alpha+1} E_p^{**}(0, \alpha)$$

$$\text{where } \xi = \frac{d-c}{h} \quad c = \frac{a+b}{2} \quad h = \frac{b-a}{2}$$

Now let  $\Delta$  be the one-dimensional mesh on  $I=(0, B)$ .

$$\Delta: 0 = x_0 < x_1 < x_2 < \dots < x_M = B$$

$$\text{where } x_i = Bq^{M-i}, \quad q < 1$$

$$\text{and denote } I_j = (x_{j-1}, x_j).$$

Assume that  $f = x^\beta$  and define

$$(5.8) \quad E(\beta, p, q, M) = \left( \inf_{\omega} \int_0^B (f' - \omega')^2 x dx \right)^{1/2}$$

where the infimum is taken over all functions  $\omega \in H^1(I)$  and  $\omega$  is a polynomial of degree  $p$  on  $I_i$ ,  $i = 1, \dots, M$  and  $\omega(x_i) = f(x_i)$ .

We now have

Theorem 3:

$$(5.9) \quad E(\beta, p, q, M) = C \cdot B \cdot \left[ \frac{q^{2\beta(M-1)}}{p^{4\beta}} + \frac{r^{2p}}{p^{2\beta}} \cdot \frac{(1-q)^{2\beta(M-1)}(1-q)q^{\beta-1}}{1-q^{2\beta}} \right]^{1/2}$$

where  $D_1(\beta) \leq C \leq D_2(\beta)$  with  $D_i$  independent of  $M$  and  $p$ .

Proof:

We will approximate  $f' = \beta x^{\beta-1}$  by polynomials of degree  $(p-1)$  on every  $I_j$ . Then by integration we construct  $\omega \in H^1(I)$ .

On  $I_j$ ,  $j > 1$  we use lemma 3 and set

$$a = x_{j-1} = Bq^{M+1-j} ; \quad b = x_j = Bq^{M-j}$$

$$h = Bq^{M-j} \frac{1-q}{2} ; \quad c = Bq^{M-j} \frac{1+q}{2}$$

$$\xi = -\frac{c}{h} = \frac{1+q}{1-q} ; \quad r = \frac{1}{|\xi| + \sqrt{\xi^2 - 1}} = \frac{1-q}{(1+q) + 2\sqrt{q}} = \frac{1-\sqrt{q}}{1+\sqrt{q}}$$

We get

$$\begin{aligned} E(x_{j-1}, x_j, 0, \beta-1) &= C(\beta) \frac{(1-r^2)^{\beta-1}}{(2r)^{\beta-1}} \frac{r^p}{p^\beta} B^{(\beta-\frac{1}{2})} q^{(M-j)(\beta-\frac{1}{2})} \left(\frac{1-q}{2}\right)^{\beta-\frac{1}{2}} \\ &= C(\beta) \frac{1}{2^{\frac{1}{2}}} (1-q)^{\frac{1}{2}} q^{\frac{\beta-1}{2}} \frac{r^p}{p^\beta} B^{\beta-\frac{1}{2}} q^{(M-j)(\beta-\frac{1}{2})} \end{aligned}$$

On  $I_1$  we have

$$E_p^{**}(0, x_1, 0, \beta-1) = C(\beta) B^\beta q^{(M-1)\beta} \left(\frac{1-q}{2}\right)^\beta \frac{2^\beta}{p^{2\beta}}$$

Hence  $E(\beta, p, q, M) = C(\beta) \left[ (1-q) q^{\beta-1} \frac{r^{2p}}{p^{2\beta}} B^{2\beta-1} \right]$ .

$$\begin{aligned} & \sum_{j=2}^M q^{2(M-j)(\beta-1)} \cdot B \cdot q^{(M-j)} + B^{2\beta} q^{2\beta(M-1)} \left(\frac{1-q}{2}\right)^{2\beta} \frac{2^{2\beta}}{p^{4\beta}} \Big]^{1/2} \\ & \approx C(\beta) B^\beta \left[ \frac{q^{2\beta(M-1)}}{p^{4\beta}} + \frac{r^{2p}}{p^{2\beta}} \frac{(1-q^{2\beta(M-1)})(1-q)q^{\beta-1}}{1-q^{2\beta}} \right]^{1/2} \end{aligned}$$

where  $C(\beta)$  is independent of  $M$  and  $p$  and is (mildly) dependent on  $q$ .

We are now able to return to the error in the critical element  $e_i$  having a vertex in the vertex  $A_i$  of the polygon. (see Fig. (5.1).)

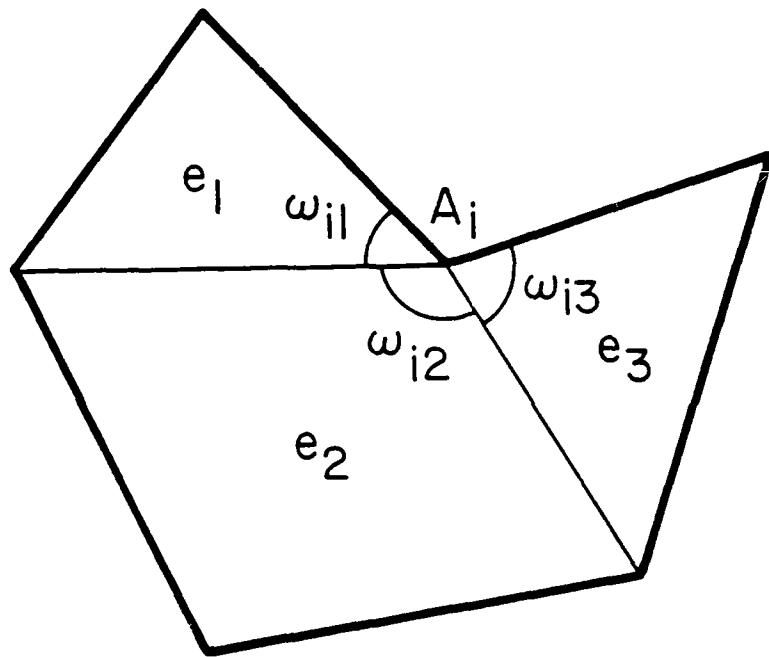


Figure 5.1. Scheme of elements at vertex  $A_i$ .

We will assume that the solution to be approximated is

$$K_1 r^{\alpha_1} \phi_1(\theta) + K_2 r^{\alpha_2} \phi_2(\theta)$$

and that the critical element is divided into  $M-1$  layers by a geometrical mesh. Furthermore, let

$$(5.10) \quad \rho^2 = C \cdot \omega_i \sum_{i=1}^2 K_i^2 E^2(\alpha_i, p, q, M) \cdot B_i^{\alpha_i}$$

$\omega_i$  is the angle of the critical element at  $A_i$  (see Fig. (5.1).) and  $B_i$  is the radius of the smallest circle centered at  $A_i$  covering the element.

The expression (5.10) is an error predictor for the error in the energy norm. It will be seen in the numerical examples that  $\rho$  reliably gives all characteristics of the error behaviour in dependence of  $M$ , i.e., the number of layers.

The value  $q$  will be chosen of the order .15 as stated above, which is the optimal value for one-dimensional problems.  $C(\beta)$  in (5.10) is not critical for our purposes because it ranges only over a small interval.

## 5.2 Optimization of the number of layers in the critical elements

As has been said, we will assume that the decisive areas for accuracy are the critical elements which have to be refined. Hence we will assume that the only error is in these elements and that the stress intensity factors are known.

For given  $p$  and number of layers  $n_i = M_i - 1$  in the  $t(i)$  elements adjacent to the critical vertices  $A_i$ ,  $i=1, \dots, s$  we get from (5.10) the error predictor for the error in the whole domain  $\Omega$

$$(5.11) \quad \rho^2(p, M_i) = C \cdot \sum_{i=1}^s \sum_{j=1}^{t(i)} w_{i,j} \sum_{\ell=1}^2 K_{\ell,i}^2 E^2(\alpha_{\ell,i}, p, q, M_i) B_{i,j}^{\alpha_{\ell,i}}$$

where  $K_{\ell,i}$  are the (known) stress intensity factors in the vertex  $A_i$  and  $\alpha_{\ell,i}$  the exponents of the singular functions in (2.4).

The number of degrees of freedom over all the refined mesh will be denoted by

$$N(p, M_i, i=1, \dots, s) .$$

Assuming that the computational work is a function of  $N$  only, we can formulate the following optimization problem:

Given an upper bound  $N_0$  of the total number of degrees of freedom, find  $p$  and  $M_i$  so that it minimizes (5.11) under the constraint

$$N(p, M_i, i=1, \dots, s) \leq N_0 .$$

For practical purposes,  $p$  can be restricted to a maximum degree,  $p_{\max}$ , and it is also reasonable to restrict the number of layers to some  $M_{\max}$ . A possible choice for the maximally allowed number of layers is  $M_{\max} = 2p_{\max}$  as it was shown in [8] that the optimal combination of  $M$  and  $p$  in the case of a single crack-tip singularity is given by  $M = p$ .

With these restrictions the optimization problem is finite and various methods for constrained, finite optimization could be used. Yet it turns out that (5.11) often has many local extrema so that the exact optimum will be hard to find. Nevertheless it is not necessary to find the minimum exactly because various simplifications have been made in obtaining (5.11). In all our test examples, the following simple heuristic algorithm gives nearly optimal results with only minor computational effort.

Perform steps 1 to 5 for  $p=1, \dots, p_{\max}$ :

Step 1: Choose as starting distribution  $M_i = M_{\max}$ ,  $i=1, \dots, s$

Step 2: Compute  $\rho(p, M_i)$ ,  $i = 1, \dots, s$  and

$N(p, M_i)$ ,  $i = 1, \dots, s$ .

Perform steps 3 and 4 until  $N \leq N_0$ .

Step 3: Find  $j \in \{1, \dots, s\}$  with the following property:

If one layer is removed at  $A_j$ , the quotient of the change of the total number of degrees of freedom and the change in the error predictor (5.11) is maximal.

Step 4: Remove one layer at  $A_j$ , update  $\rho(p, M_i)$ ,  $i=1, \dots, s$  and  $N(p, M_i)$ .

Step 5:  $\{M_i\}$ ,  $i=1, \dots, s$  is now a locally optimal distribution for the specified  $p$ -degree.

In practice it is often useful to compute the error predictor as a function of  $N_0$ , for optimal distributions of the degree  $p$  and the number of layers, i.e., to perform a sequence of optimization processes.

### 5.3 Prediction of the total error

Because of the constant  $C$  in (5.10) the error predictor estimates the total error in the energy norm only up to a constant, which has still to be determined. As has been shown in [8], the error behaviour for optimal meshes is of the type  $\|e\|_E \approx e^{-\beta N^{1/3}}$ . Hence it is advantageous to visualize the error in the scale  $\lg\|e\| \times N^{1/3}$ . The error predictor gives good qualitative characterization of the total error (including the errors in noncritical elements), if it will be properly scaled by a shift of  $\lg(\|e\|)$ . This adjustment determines the proper constant  $C$  in (5.10) and will be called

calibration. The main idea is to calibrate the error predictor (i.e., the curve  $\lg(\|e\|) \propto N^{1/3}$ ) so that for small  $p$  the error predictor is close to the actual error. This can be done by various approaches. For example, we can compute the solutions for  $p=1$  and  $p=2$  or  $p=1,2,3$  on the basic mesh and determine the approximate error by extrapolation, assuming that

$$(5.12) \quad \|e\|_E = C N^{-\delta}$$

for the  $p$ -version on a fixed mesh where  $\delta$  depends on the smoothness of the solution. Experience shows that for low  $p$  and coarse meshes  $\delta = .70$  to  $\delta = .75$  gives reasonable results in most practical cases.

## 6. NUMERICAL EXAMPLES

In two numerical examples we will show the steps of the expert system as described in section 4. The first example will have only one singular point, showing the basic principles of the error prediction. The second example will have 9 singular points and show that superior accuracy can be obtained in practical problems.

### 6.1. Problem 1: The L-shaped Domain

Figure 6.1 shows the basic mesh of three elements constructed in step a) of section 4. The domain is loaded by tractions so that the exact solution shows mode 1 stress distribution given, for example, in [18]. The solution has singular behaviour at the reentrant corner.

As each of the three elements is adjacent to the reentrant corner, in step b) each element is expected to be critical.

Step c) extracts critical data about the solution of the problem. In order to get sufficiently accurate data for an extrapolation of the exact energy (see section 5.3) and an accurate relation of the two stress intensity factors  $K_1$  and  $K_2$  at the reentrant corner, one circular refinement layer was added at the reentrant corner (Fig. (6.2)). On this mesh a finite element computation was performed for  $p=1,2,3$ . This crude computation yielded for  $p=2$  approximate stress intensity factors  $K_1 = 0.91$ ,  $K_2 = 0$  (the exact values for this example are:  $K_1 = 1$  and  $K_2 = 0$ ). The approximate energy of the three solutions was  $U_1 = 3.8288279$ ,  $U_2 = 4.0975754$ ,  $U_3 = 4.134477$ . The stress intensity exponents are only geometrical data and can be computed exactly. They are given by  $\alpha_1 = 0.544484$  and  $\alpha_2 = 0.908529$ .

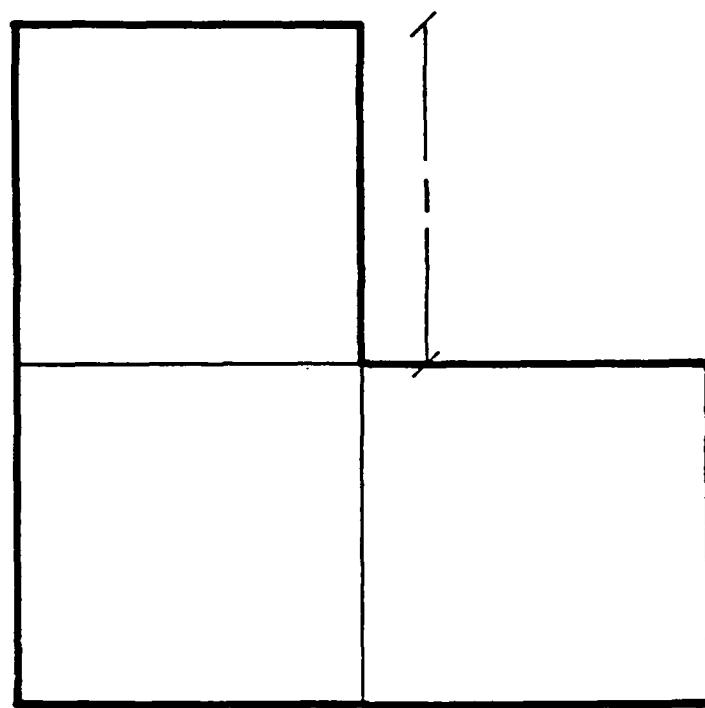


Figure (6.1). L-domain, basic mesh

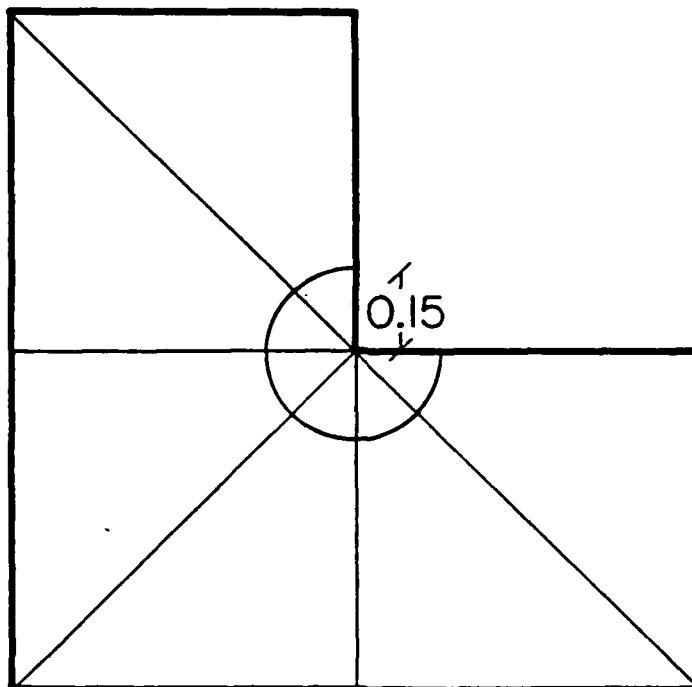


Figure (6.2) L-domain, elementary mesh for crude computation.

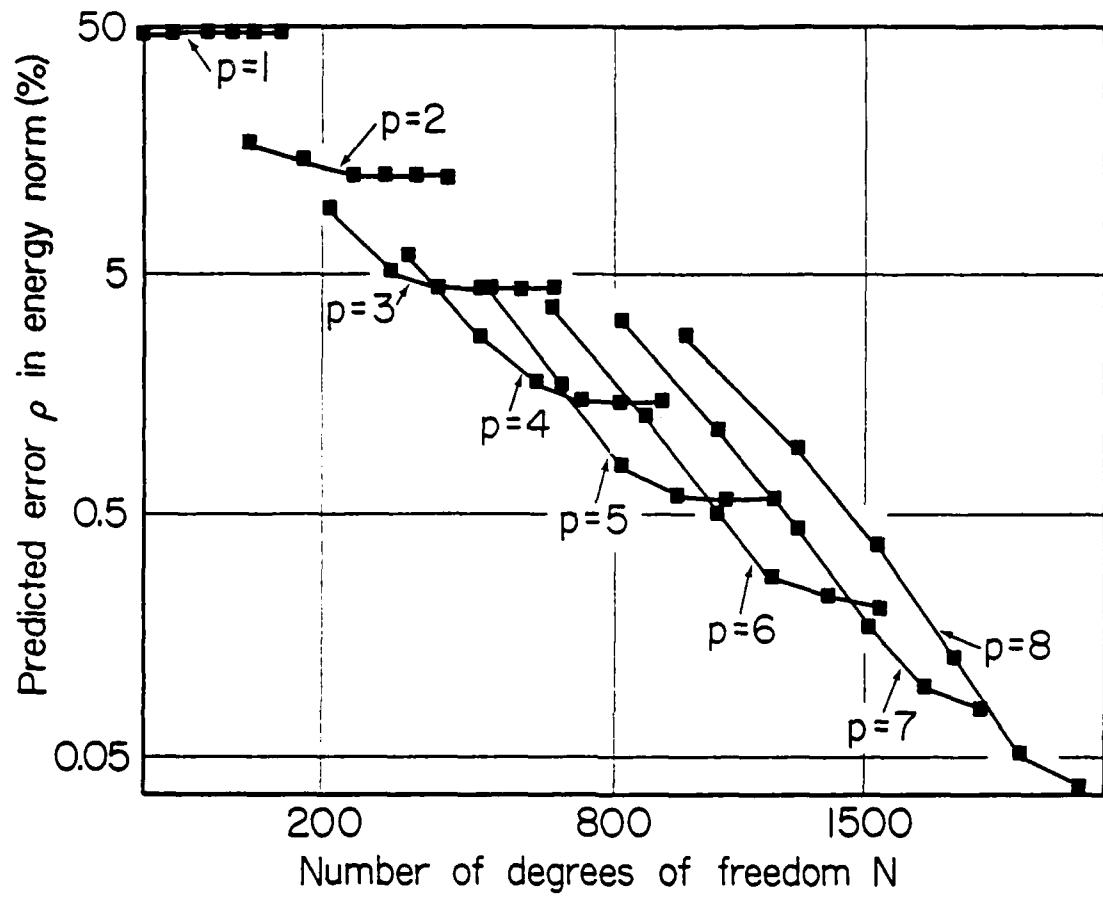


Figure 6.3. L-domain. Predicted error

Steps d) and e) predict the performance of various optimal meshed and p-degrees and present this prediction graphically to the user. Out of the computed energy of the solutions for  $p=1,2$ , the exact energy is extrapolated by (5.12) with  $\delta = .75$  to  $\hat{U} = 4.1646802$  which estimates well the exact strain energy of  $U_{ex} = 4.1545442 \cdot \hat{U}$  served to calibrate the predicted error in the energy norm. Figure (6.3) shows the error prediction as it is presented to the user for  $p=1,\dots,8$  and  $n=2,\dots,7$ .

In steps f) and g) one particular mesh-degree combination would be constructed automatically after the user's decision and the finite element computation would be performed. In order to show the performance of the error prediction, we compare Fig. (6.3) with the actually computed solutions on the combinations of mesh and polynomial degree. The numerical results are taken from [19]. Figure (6.4) gives the real error in the energy norm. All mesh-degree combinations, which were predicted to be optimal for a specific number of degrees of freedom, are marked with circles in Fig. (6.4). It shows that each of these meshes is on the lower left envelope, which means that the prediction of the optimality was right in each case.

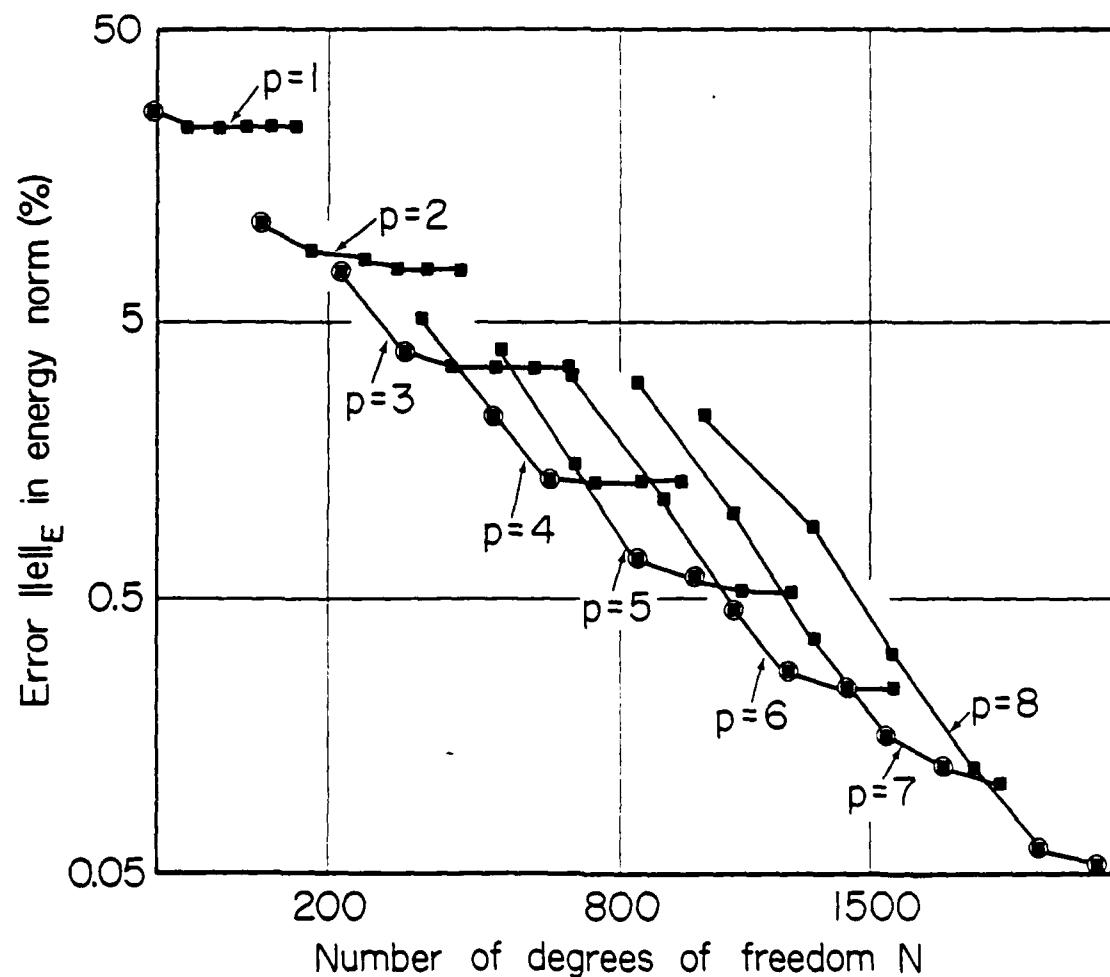


Figure 6.4. L-domain. Exact error.

m/p	NDOF	e%	p%	θ
2/2	79	11.71	16.64	1.42
2/3	143	6.95	8.52	1.22
3/3	209	3.93	4.94	1.26
3/4	335	2.14	2.46	1.12
4/4	439	1.43	1.64	1.15
4/5	651	0.735	0.772	1.05
5/5	805	0.585	0.577	0.99
4/6	911	0.476	0.495	1.04
5/6	1127	0.286	0.259	0.91
6/6	1343	0.253	0.211	0.83
5/7	1509	0.168	0.156	0.93
6/7	1799	0.121	0.091	0.75
6/8	2327	0.066	0.051	0.77
7/8	2703	0.054	0.033	0.61

Table 6.1: Actual error, predicted error and effectivity index  
for L-shaped domain.

The slope of the curve joining the optimal mesh-degree combinations, i.e., the envelope in Fig. 6.4 is a straight line in the  $\log-N^{1/3}$ -scale, which shows that exponential rate of convergence is obtained by this feedback process. Finally in table (6.1) the efficiency index of the error predictor  $\theta = \frac{\rho}{|e|}$  is given for all optimal meshes.  $\theta$  ranges from 0.61 to 1.42 which shows that the relative error is reasonably estimated by the error prediction.

We performed the same computations on geometrical meshes of the type shown in Fig. 3.1 with elements with only straight edges. Qualitative and quantitative behaviour of error prediction and actual error was very similar to the meshes with circular refinement around the singularity, so that these results are not reported in detail.

#### 6.2. Problem 2: The Wrench

Figure 6.5 shows the domain of computation and the loads of the second problem. The domain has 9 reentrant corners (including the two tips of the crack interior to the domain). Constant traction is applied along the edge CD and a symmetry boundary condition is imposed along AB. Isotropic material with  $E = 3E8$  psi and  $\nu = .3$  was assumed.

Figure 6.6 shows the basic mesh of this problem as it was constructed interactively by a modification of a mesh generator of MODULEF [20]. The basic mesh is as coarse as possible, only modelling the geometry of the problem (part a) in section 4))

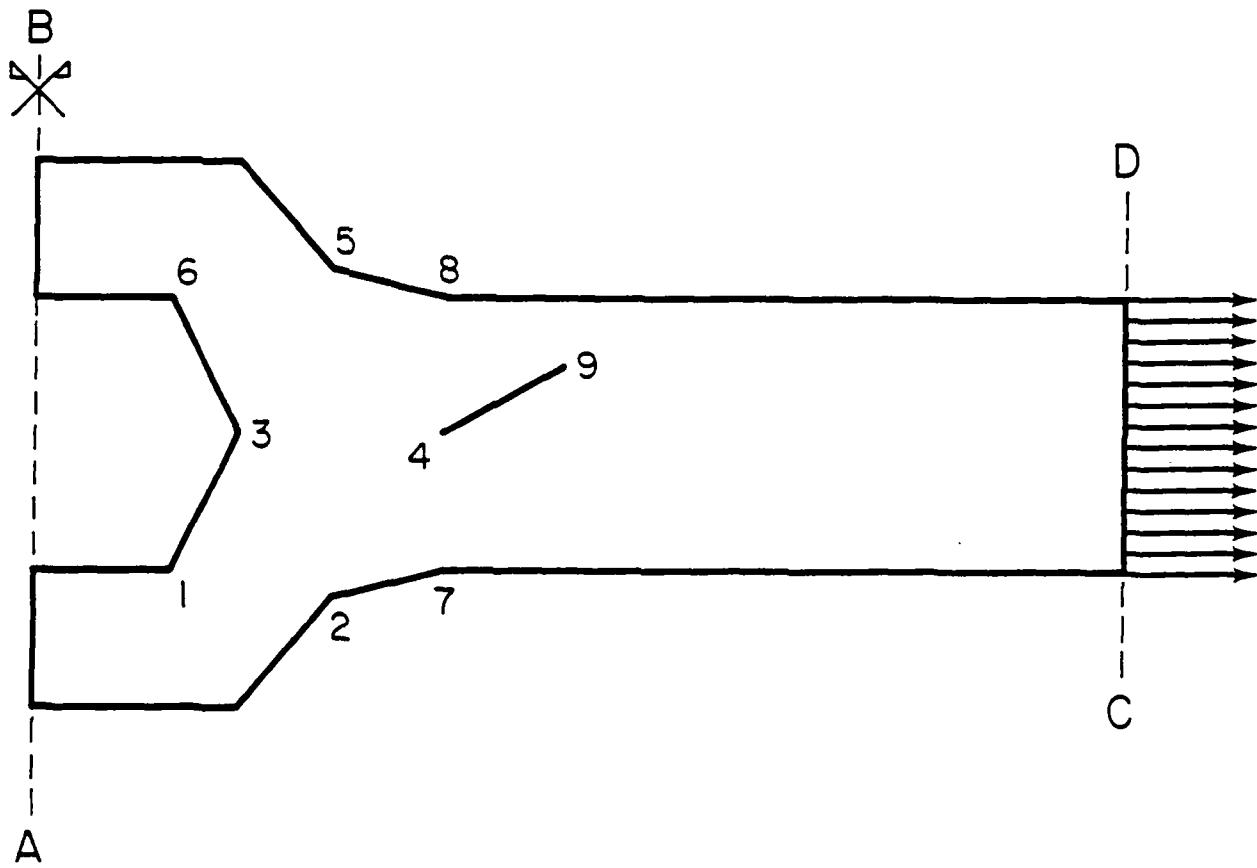


Figure 6.5. Domain and load for problem 2.

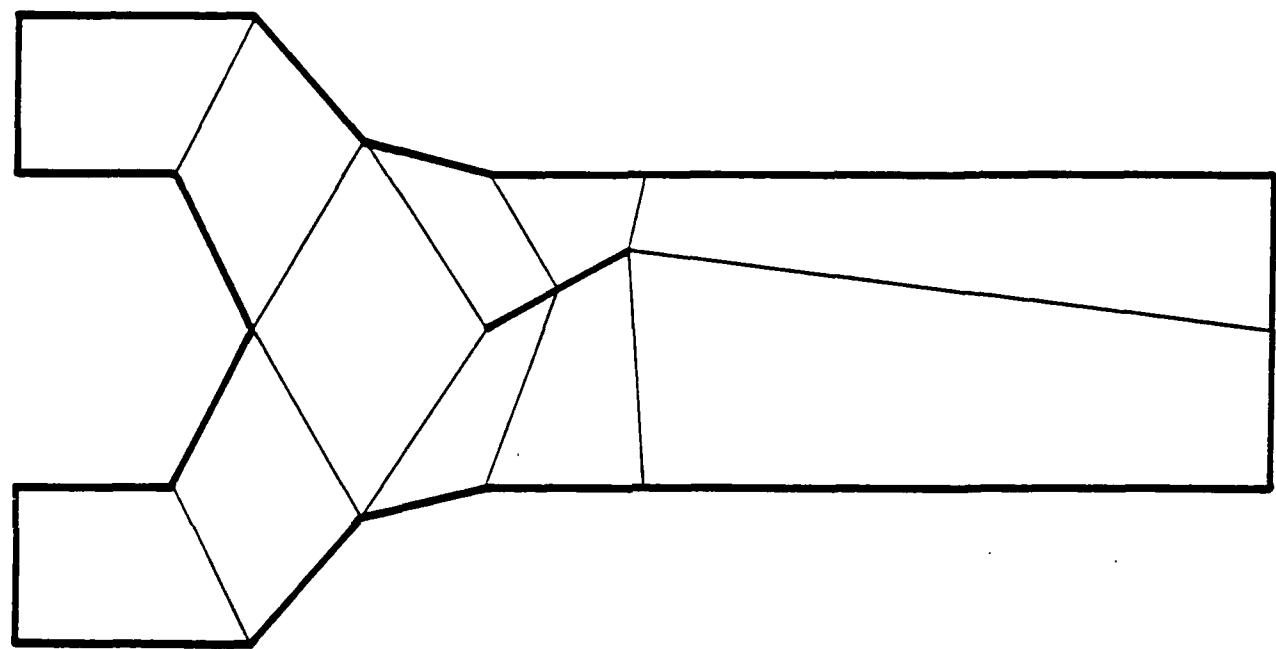


Figure 6.6. Basic mesh for problem 2.

In the next step of the analysis (part b) the decision about critical and noncritical elements is made. Again, each element is critical as each has at least one reentrant corner as node.

Step c) extracts critical data about the solution. As was shown in section 5, the stress intensity factors  $K_{j,i}$  and exponents  $\alpha_{j,i}$  of the singular functions provide means to predict the error in the energy norm for all combinations of (geometric) mesh and polynomial degree  $p$ . For a sufficiently accurate extraction of stress intensity factors it is necessary to have a mesh such that at most one singular point is in each element. Therefore out of the basic mesh an elementary mesh is constructed automatically, Fig. 6.7. Table 6.2 gives the stress intensity factors  $K_{j,i}$ ,  $j=1,2$  for  $p=2$ . As reference solution, "exact" stress intensity factors are shown, which were obtained on a strongly refined mesh with  $p=5$  and 2985 degrees of freedom. It can be seen that all stress intensity factors are within a relative error of about 15% compared to the largest stress intensity factor ( $K_{2,1} = -.157154 \times 10^4$  at point 1). The strain energy on the elementary mesh for  $p=1$  is  $U_1 = 0.0323163736$  and for  $p=2$   $U_2 = 0.0357676705$ . Using  $\delta = .75$  in (5.12), the estimated exact energy is  $\hat{U} = 0.03673856$ , which yields an estimated error in the energy norm of 14.4% for the elementary mesh with  $p=3$  ( $U_3 = 0.0359766585$ ). This value is used to calibrate the error predictor (see 5.10).

Step d) predicts the error for  $p=2,3,4$  and 5 for various combinations and optimizes mesh and polynomial degree for a sequence of  $N_0$  as described in section 5.2. The result of this optimization is shown in Fig. 6.8 and table 6.3 as it is presented to the user in step e). Each point in Fig. 6.8 corresponds to a certain layer distribution given in table 6.3. Again, the next step in practice would be the decision of the user on a particular mesh-degree

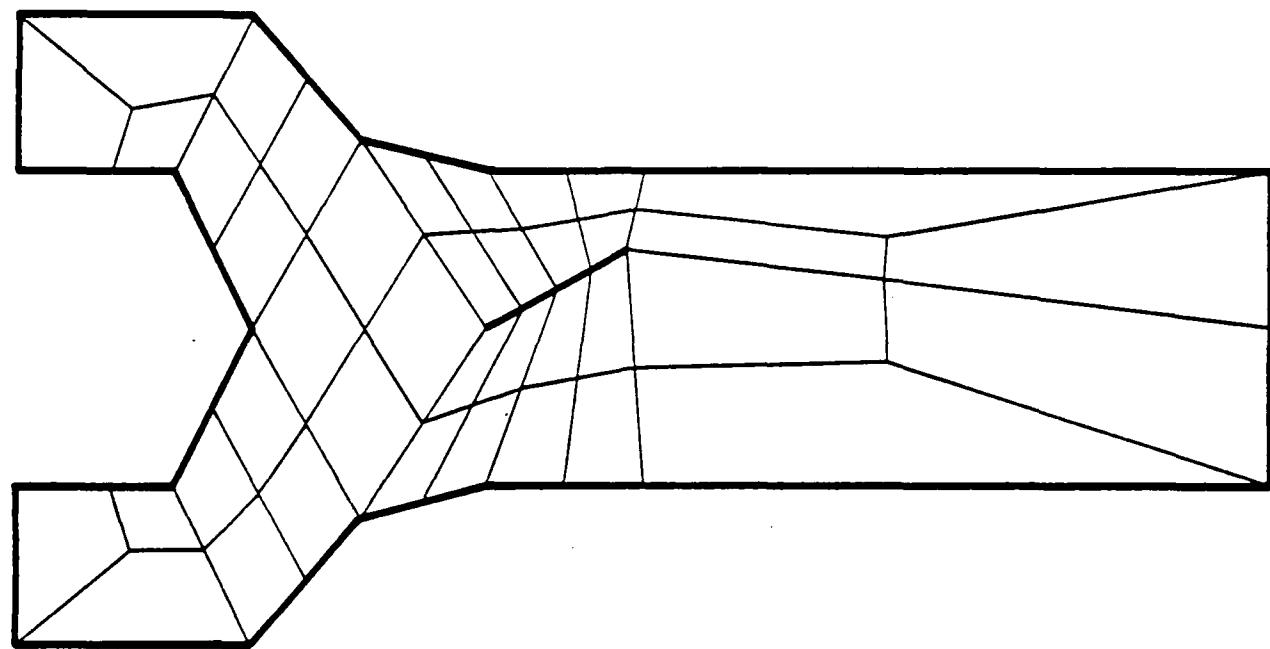


Figure 6.7. Mesh for crude computation of problem 2.

point	factors	basic mesh	refined mesh	relative error (%)
1	$K_I$	0.359377 E3	0.377353 E3	1.2
	$K_{II}$	-0.155661 E4	-0.157154 E4	0.9
2	$K_I$	0.551988 E3	0.574465 E3	1.4
	$K_{II}$	-0.992989 E2	-0.109476 E3	0.6
3	$K_I$	-0.536166 E3	-0.534692 E3	0.0
	$K_{II}$	-0.205327 E2	-0.983101 E1	0.6
4	$K_I$	0.299322 E3	0.539827 E3	15.3
	$K_{II}$	-0.943211 E2	-0.168955 E3	4.7
5	$K_I$	0.561447 E3	0.580995 E3	1.2
	$K_{II}$	0.127882 E3	0.128434 E3	0.0
6	$K_I$	0.358611 E3	0.374829 E3	1.0
	$K_{II}$	0.156916 E4	0.155843 E4	0.6
7	$K_I$	0.325866 E3	0.311223 E3	0.9
	$K_{II}$	0.115326 E3	0.753421 E2	2.5
8	$K_I$	0.420292 E3	0.437345 E3	1.1
	$K_{II}$	-0.295953 E2	0.224532 E2	3.3
9	$K_I$	0.204060 E3	0.234956 E3	2.0
	$K_{II}$	-0.219939 E3	-0.218943 E3	0.0

Table 6.2. Stress intensity factors from basic mesh ( $p=2$ ) and refined mesh ( $p=5$ ).

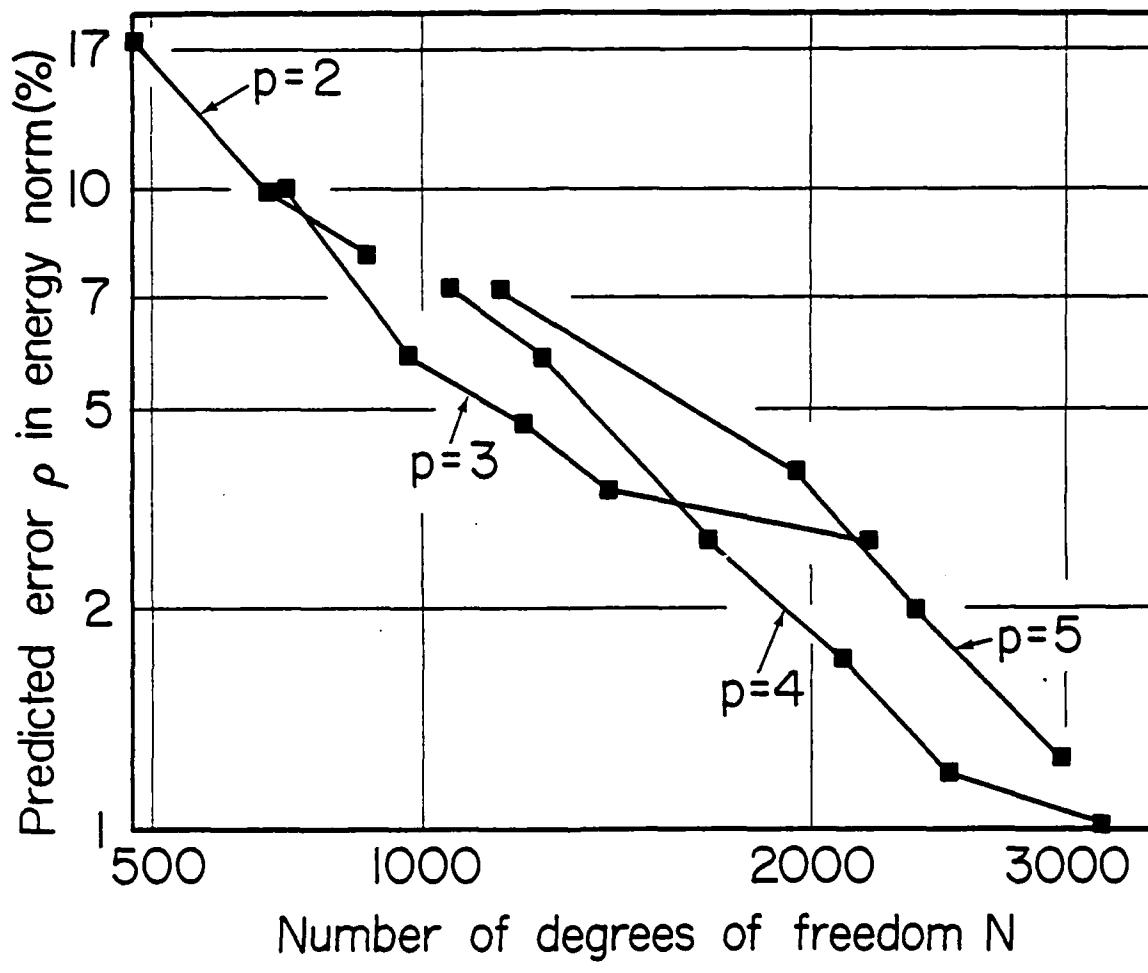


Figure 6.8. Predicted error for problem 2.

Polynomial degree p	Total number of degrees of freedom	Element layers of point								
		1	2	3	4	5	6	7	8	9
2	488	2	2	2	1	2	2	1	1	1
2	692	3	2	2	2	2	3	2	2	2
2	864	3	2	3	3	3	3	2	2	3
3	703	2	1	2	1	1	2	1	2	1
3	941	2	2	2	2	2	2	1	1	2
3	1191	3	2	2	3	2	3	2	2	2
3	1409	3	2	3	3	3	3	2	2	3
3	1673	3	3	3	4	3	3	2	3	4
3	2187	5	4	4	4	4	5	3	3	5
4	1021	2	1	1	2	1	2	1	1	1
4	1229	2	1	2	2	2	2	1	1	1
4	1685	2	2	2	2	2	3	2	2	2
4	2101	3	2	2	3	3	3	2	2	3
4	2517	3	3	4	4	3	3	2	2	3
4	3192	4	3	4	5	3	4	3	3	5
5	1127	1	1	1	1	1	1	1	1	1
5	1955	2	1	2	2	1	2	2	1	2
5	2369	3	2	2	2	2	2	1	2	2
5	2985	2	2	3	3	2	3	2	2	3

Table 6.3. Distribution of layers

combination. In contrast to that, we performed here finite element computations for all meshes and degrees as specified in table 6.3. The refined meshes were constructed automatically by the mesh generator. Figure 6.9 shows one example of a refined mesh, corresponding to the combination with  $p=5$  and 2985 degrees of freedom in table 6.3. Figure 6.10 shows the real error for all computations. We estimated an exact energy from extrapolation on an extremely refined mesh and high polynomial degree to  $U_{EX} = 0.0367809$ , which is very close to the estimated energy after the crude computation in step c). Comparing Figs. 6.8 and 6.10, it can be seen that the real error for a certain  $p$ -degree levels off earlier as the predicted error. This seems to be due to the fact that the error predictor neglects the error in "smooth" parts of the solution which are still significant for low  $p$ -degrees.

However, 8 of 10 optimal points (points at the lower left envelope in Fig. 6.8 which are marked by circles in Fig. 6.10) turn out to be really the best meshes for the specified number of degrees of freedom, the remaining two meshes having only mildly larger error than the actually best combinations. It should again be mentioned that the optimal meshes constructed in the feedback process, converge with exponential rate, which shows that the feedback really yields optimal mesh design. Finally table (6.4) gives the efficiency index  $\theta = \frac{T_e}{T_E}$  for all the meshes which is in the range of 0.43 to 1.35.

By this feedback process it is possible to construct a mesh which yields an accuracy in the energy norm of about 2% for 3000 degrees of freedom. Extrapolation from the elementary mesh shows that this accuracy could be obtained with a quasiuniform mesh using linear elements (assumed convergence rate  $\delta = .25$ ) with about  $1E7$  degrees of freedom. An adaptively constructed mesh, using the h-version with linear elements (convergence rate  $\delta = .5$ ) should yield this accuracy with about 34,000 degrees of freedom. About the

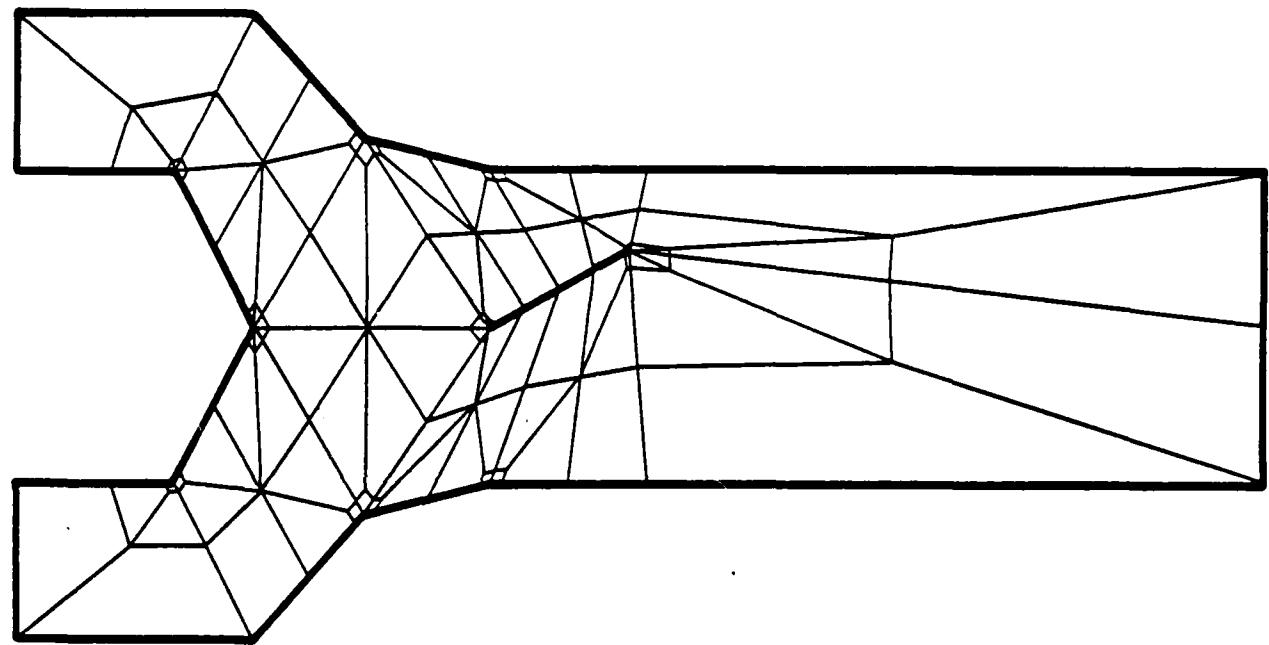


Figure 6.9. Refined mesh for  $p=5$ , 2985 NDOF.

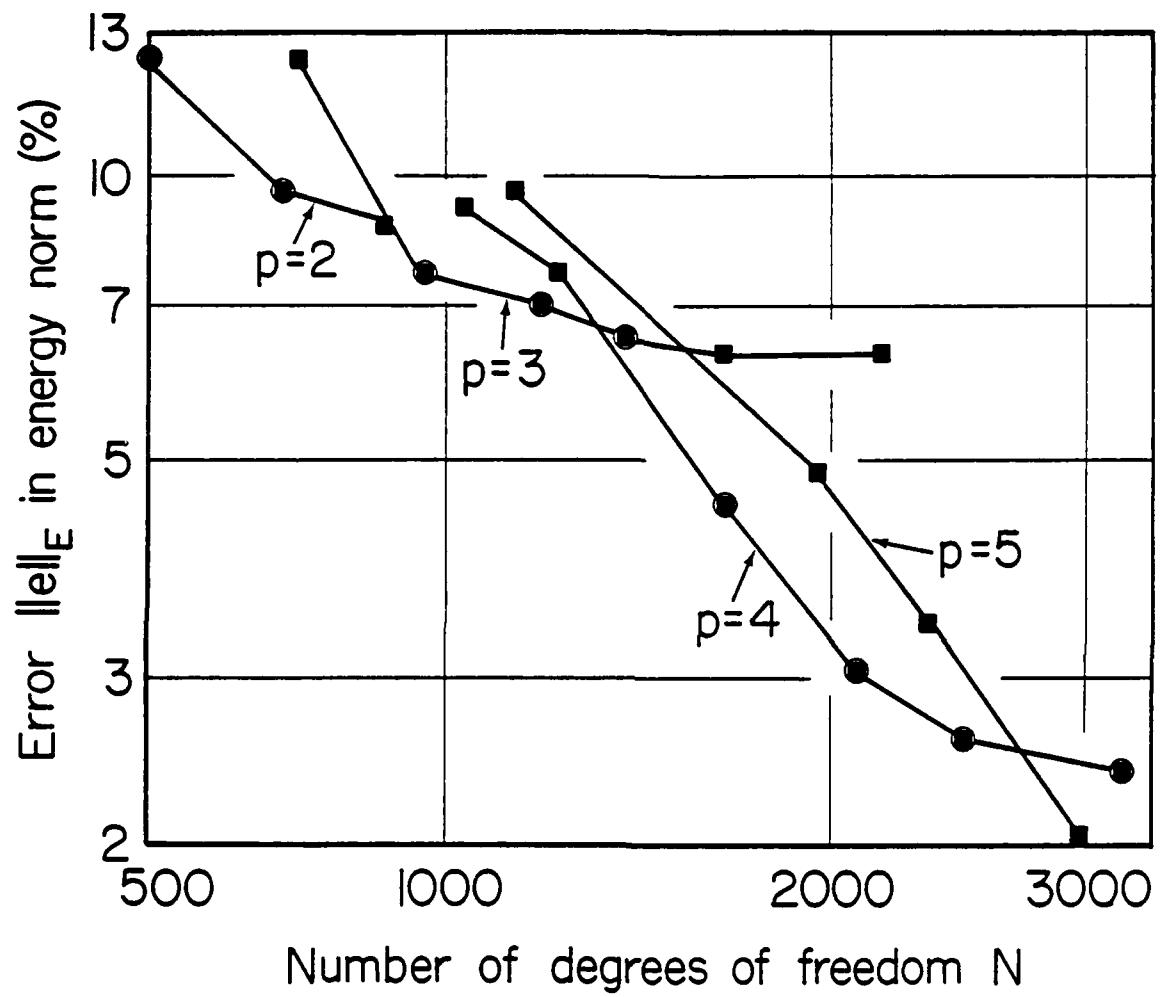


Figure 6.10. Actual error for problem 2.

same amount of dofs would be necessary for a pure p-version on the elementary mesh. From these estimations it can be seen that about one order of magnitude in the number of degrees of freedom is gained compared to an adaptive h-version or a pure p-version. So, roughly two orders-of-magnitude in storage and computational time is gained if high accuracy solutions with an error in the range of 1 to 2% are desired.

The additional cost for the feedback is negligible. On an Apollo 420 the overall time for the computation on the elementary mesh ( $p=1,2,3$ ) together with graphical mesh generation, the sequence of optimizations of section 5.2 for  $p=2,3,4,5$ ,  $N_0$  from 400 to 4000 in steps of 200 degrees of freedom and the construction of a refined mesh with 2985 dofs was less than 800 CPU sec, whereas the final computation on the refined mesh with  $p=5$  took more than 4000 CPU seconds.

Moreover, it should be mentioned that the human time is completely independent of the desired accuracy, a situation which is totally different from conventional finite element analysis, where higher accuracy can only be achieved by time consuming construction of refined meshes. Only the basic mesh has to be designed by human interference and, after this, only decisions about the progress of the analysis have to be made.

Polynomial degrees p	Total number of degrees of freedom	$\rho(\%)$	$ e _E(\%)$	$\theta$
2	488	17.21	12.74	1.35
2	692	9.95	9.42	1.06
2	864	8.07	8.56	0.94
3	703	9.99	12.54	0.80
3	941	5.58	7.71	0.72
3	1191	4.32	7.05	0.61
3	1409	3.34	6.38	0.52
3	1673	2.92	6.18	0.47
3	2187	2.87	6.17	0.47
4	1021	7.01	8.83	0.79
4	1229	5.50	7.46	0.74
4	1685	2.86	4.39	0.65
4	2101	1.85	2.99	0.62
4	2517	1.26	2.53	0.50
4	3192	1.00	2.30	0.43
5	1127	7.00	9.33	0.75
5	1955	3.67	4.71	0.78
5	2369	2.24	3.30	0.68
5	2985	1.29	1.94	0.66

Table 6.4. Estimated error, exact error and effectivity index.

## 7. CONCLUSIONS

The main ideas of a knowledge-based system advising the analyst how to design the mesh and degree distribution for the hp-version of the finite element method have been presented. The system includes preliminary analysis of the problems and, based on this analysis, it advises the analyst how to get the prescribed accuracy for the lowest cost. Although the ideas are restricted to a particular class of problem one can expect that ideas of the kind we presented can be extended to more general cases and combined with other CAD tools, solid modellers, etc., to a larger scale expert system.

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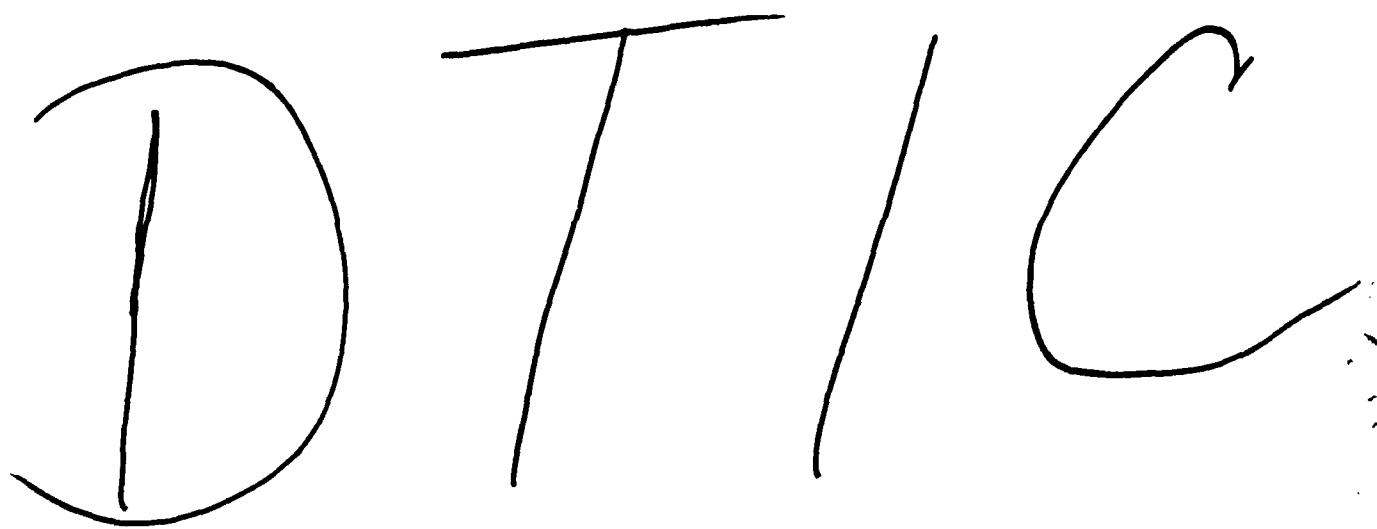
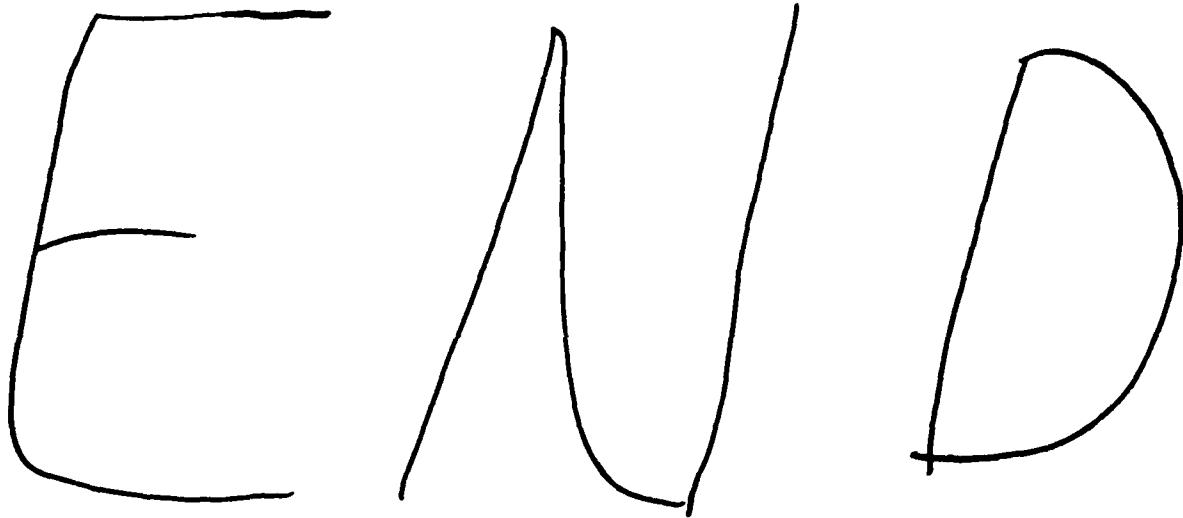
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Further information may be obtained from Professor I. Babuska, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.



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